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# Justification and failure of the nonlinear Schrödinger equation in case of non-trivial quadratic resonances

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**Abstract**

The nonlinear Schrödinger (NLS) equation can be derived as an amplitude equation describing slow modulations in time and space of an underlying spatially and temporarily oscillating wave packet. The purpose of this paper is to prove estimates, between the formal approximation, obtained via the NLS equation, and true solutions of the original system in case of non-trivial quadratic resonances. It turns out that the approximation property (APP) holds if the approximation is stable in the system for the three-wave interaction (TWI) associated to the resonance. We construct a counterexample showing that the NLS equation can fail to approximate the original system if instability occurs for the approximation in the TWI system. In the unstable case we give some arguments why the validity of the APP can be expected for spatially localized solutions and why it cannot be expected for non-localized solutions. Although, we restrict ourselves to a nonlinear wave equation as original system we believe that the results hold in more general situations, too.


*Keywords:* NLS equation; Approximation; Validity

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## 1. Introduction

The Nonlinear Schrödinger (NLS) equation can be derived as an amplitude equation describing slow modulations in time and space of an underlying spatially and temporarily oscillating wave packet in nonlinear hyperbolic systems, as Maxwell's equations in nonlinear optics [AS81] or the equations describing surface water waves [Za68]. For general quasilinear systems under the validity of some non-resonance condition on the linear dispersion relation Kalyakin [Kal88] proved estimates between the formal approximation obtained via the NLS equation and true solutions of the original nonlinear hyperbolic system. It is the purpose of this paper to prove such estimates in case that this non-resonance condition is not satisfied, i.e. in case that the original system possesses non-trivial quadratic resonances. It turns out that the approximation property (APP) holds, if the approximation is stable in the system for the three-wave interaction (TWI) associated to the resonance. The unstable case is less clear. In case of spatially periodic solutions, we construct a counterexample showing that the NLS equation can fail to approximate the original system if instability occurs for the approximation in the TWI system. This means that unstable non-trivial quadratic resonances are able to destroy solutions in NLS form even before the beginning of the natural time scale of the NLS equation, i.e. the NLS equation makes then wrong predictions about the behavior of the original system. However, we give some arguments why the validity of the APP in the unstable case still can be expected for spatially localized solutions and in contrast why it cannot be expected for non-localized solutions. Although, we restrict ourselves to a nonlinear wave equation as original system we believe that the results hold in more general situations, too.

In order to be more precise let us consider throughout this paper the model

$$\partial_t^2 u = \partial_x^2 u - u - \alpha_{\rm r}\partial_x^4 u + \alpha_{\rm q} u^2 - \alpha_{\rm c} u^3 \tag{1}$$

with $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geqslant 0$ and parameters $\alpha_q, \alpha_r, \alpha_c \in \{0,1\}$, with $\alpha_c = 1$ throughout the introduction, as original system. The linearized system possesses solutions $e^{i(kx+\omega t)}$, where the temporal wavenumber $\omega \in \mathbb{R}$ and the spatial wavenumber $k \in \mathbb{R}$ are related through the linear dispersion relation

$$\omega^2 = 1 + k^2 + \alpha_r k^4. \tag{2}$$

The NLS equation allows us to describe slow modulations in time and space of a wave train $e^{i(k_0 x + \omega_0 t)}$ in (1), where $k = k_0$ and $\omega = \omega_0$ satisfy (2). It can be derived by the ansatz $u(x,t) = \varepsilon \psi_{\mathrm{NLS}}(\varepsilon, x, t) + \mathcal{O}(\varepsilon^2)$ where

$$\varepsilon \psi_{\mathrm{NLS}}(\varepsilon, x, t) = \varepsilon A\left(\varepsilon(x + c_g t), \varepsilon^2 t\right) e^{i(k_0 x + \omega_0 t)} + \text{c.c.} \tag{3}$$

with $c_g = \partial_k \omega|_{k=k_0, \omega=\omega_0}$ the negative group velocity, $A = A(X,T) \in \mathbb{C}$ a complex-valued amplitude describing the envelope of the wave packet, and c.c. the complex conjugate. Inserting this into (1) and equating the coefficients in front of $\varepsilon^{j_1} e^{j_2 i(k_0 x + \omega_0 t)}$ with $j_1 \geqslant 0$, $j_2 \in \mathbb{Z}$, to zero gives the NLS equation

$$\partial_T A = i\nu_1 \partial_X^2 A + i\nu_2 A|A|^2 \tag{4}$$

with $T = \varepsilon^2 t$, $X = \varepsilon(x + c_g t)$, and coefficients

$$\begin{aligned} \nu_1 &= -\frac{1}{2}\partial_k^2 \omega|_{k=k_0,\omega=\omega_0} = \frac{1 - c_g^2 - 6k_0^2}{2\omega_0}, \\ -2i\omega_0 \nu_2 &= 3 + 4\alpha_q^2 + \frac{2\alpha_q^2}{1 + 4k_0^2 + 16\alpha_r k_0^4 - 4\omega_0^2}. \end{aligned}$$

In a number of papers estimates have been established showing that the NLS equation provides a good approximation in the following sense.

(APP) *Let* $A \in C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$ *be a solution of the NLS equation* (4) *for a* $s_A \geqslant 0$ *sufficiently big. Then there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there are solutions* $u$ *of* (1) *and* $T_1 = T_0$ (*notation for later purposes*) *with*

$$\sup_{t \in [0, T_1/\varepsilon^2]} \sup_{x \in \mathbb{R}} |u(x,t) - \varepsilon \psi_{\mathrm{NLS}}(\varepsilon, x, t)| \leqslant C\varepsilon^{3/2},$$

*where* $\varepsilon \psi_{\mathrm{NLS}}(\varepsilon, x, t)$ *has been defined in* (3).

This means that the dynamics of the NLS equation on a time interval $[0, T_0]$ can really be seen in the original system, too, since for $\varepsilon \to 0$ the error of order $\mathcal{O}(\varepsilon^{3/2})$

is much smaller than the solution and the approximation which are both of order $\mathcal{O}(\varepsilon)$ in $L^\infty$.

Such an approximation property should not be taken for granted since there already exists a number of counterexamples which show that an original system can behave completely different than predicted by the amplitude equation, although the amplitude equation has been derived in a formally correct way, cf. [Schn95]. It will be one of the highlights of this paper to prove that also the NLS equation can fail to approximate the original system.

The proof of the APP for the NLS equation (4) and our model (1), or for more general systems differs strongly in its difficulty depending on the properties of the nonlinearity and the linear dispersion relation.

The case $\alpha_{\mathrm{q}} = 0$, i.e. more general no quadratic terms in the nonlinearity, turned out to be more less trivial. A simple application of Gronwall's inequality yields the required result [KSM92].

The case $\alpha_{\mathrm{q}} \neq 0$, i.e. more general the case of quadratic terms in the nonlinearity, is much more complicated. The case $\alpha_{\mathrm{r}} = 0$ which corresponds more general to the case of *no* non-trivial quadratic resonance in the linear dispersion relation, i.e.

$$r_{m,n}(k) = |\omega_m(k) - \omega_0 - \omega_n(k - k_0)| \neq 0$$

for all $k \in \mathbb{R}$ and $m$, $n$ has been handled for general quasilinear hyperbolic systems in [Kal88], where the different $\omega$'s are defined by the solutions $\omega = \omega_\pm(k)$ of the linear dispersion relation (2). Due to the quadratic terms, in the equations for the error there are now linear terms with respect to the error (coming from the approximation) of order $\mathcal{O}(\varepsilon)$ (instead of order $\mathcal{O}(\varepsilon^2)$ in case of the cubic terms $u^3$). In order to obtain estimates on the long time scale $\mathcal{O}(1/\varepsilon^2)$ these terms have to be controlled. They turned out to be oscillatory and can be removed by some normal form transform if the non-resonance condition

$$\inf_{k \in \mathbb{R}, m, n} r_{m,n}(k) > 0 \tag{5}$$

is satisfied. After the removal again Gronwall's inequality can be applied. The paper [Kal88] also includes the situation of trivial resonances, i.e. then the quadratic terms and $r = r_{m,n}(k)$ vanish for the same wavenumber $k$. It is easy to compute that for our model (1) in the case $\alpha_{\mathrm{r}} = 0$ the eigenvalues $i\omega_\pm(k) = \pm i\sqrt{1+k^2}$ satisfy $r_{m,n}(k) \geqslant C > 0$ with a constant $C = C(k_0) \to 0$ for $k_0 \to \infty$. See Fig. 1.

In the case $\alpha_{\mathrm{r}} \neq 0$ the non-resonance condition (5) is no longer satisfied, i.e. there exist $m, n, \tilde{k}$ with $r_{m,n}(\tilde{k}) = 0$. See Fig. 2. Nevertheless, the NLS equation can be derived, and so the question occurs, does the APP also hold in case of non-trivial quadratic resonances or do the quadratic resonances destroy the NLS form (3) even before the beginning of the natural time scale of the NLS equation.

A first attempt to approach this problem has been made in [Schn98b] where a weakened approximation property, $T_1 = \mathcal{O}(1) < T_0$, has been shown under the assumption of initial conditions for the NLS equation analytic in a strip in the complex plane

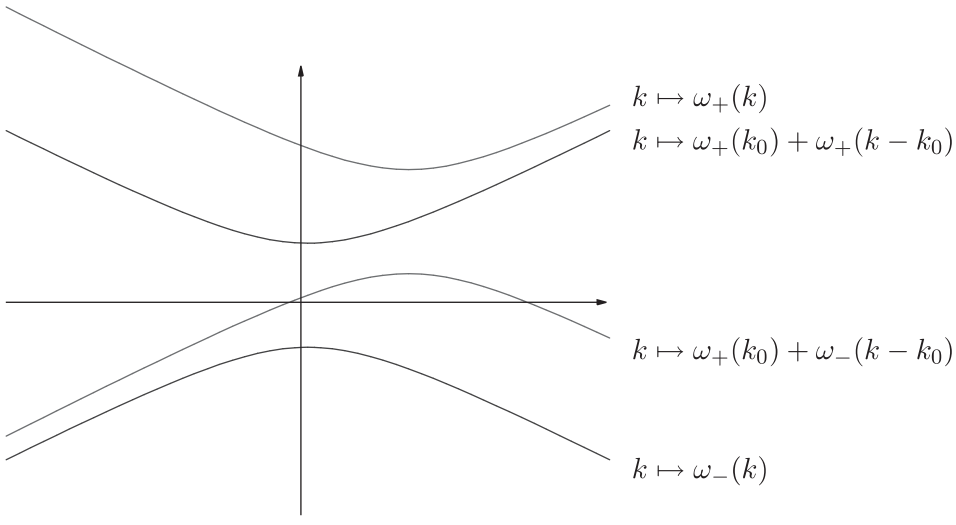


Fig. 1. Validity of the non-resonance condition in case $\alpha_r = 0$.

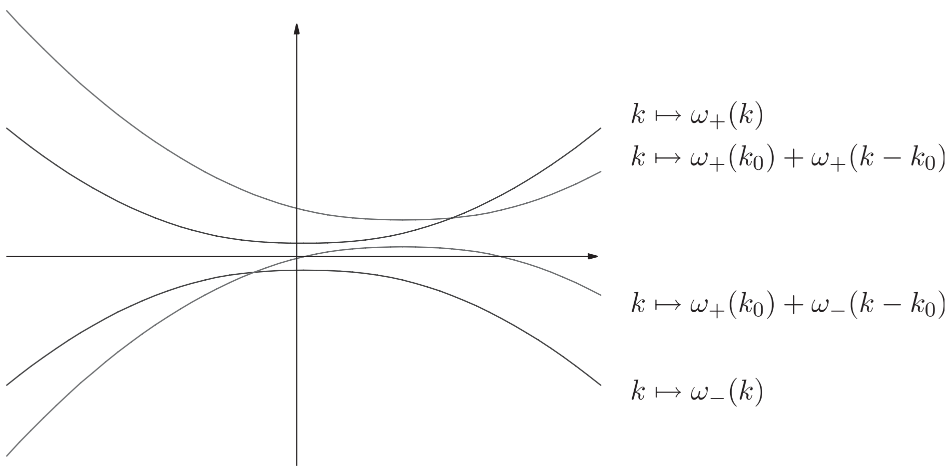


Fig. 2. Non-validity of the non-resonance condition in case $\alpha_r = 1$. Intersection points correspond to resonances.

and the assumption that the spatial wavenumbers $k$ of the resonances are bounded away from integer multiples of the basic spatial wavenumber $k_0$. The analyticity of the initial conditions for the NLS equation implies that the resonant Fourier modes are exponentially small initially. Hence, it takes a time $\mathcal{O}(1/\varepsilon^2)$ for the resonant modes to grow to a reasonable size. Multiplying exponentially small initial conditions of order $\mathcal{O}(\exp(-r_1/\varepsilon))$, for a $r_1 > 0$ fixed, with the maximal growth rate $\mathcal{O}(\exp(r_2\varepsilon t))$, for a $r_2 > 0$ fixed, of the linear evolution operator in the equation for the error shows that resonant modes stay small for all $\varepsilon^2 t \leqslant T_1 = r_1/r_2$.

The purpose of this paper is to improve this result and to show the APP with $T_1 = T_0$ in case of non-trivial quadratic resonances also for initial conditions in usual Sobolev spaces or resonant wavenumbers lying on integer multiples of the basic wavenumber $k_0$. As already said it turns out that the APP holds if the approximation is stable in the system for the TWI associated to the resonance. It can fail if instability occurs for the approximation in this system.

Fig. 2 shows that for a given $k_0 > 0$ we have at least one pair of resonances $k_2, k_3$ resonant to $k_1 = k_0$ satisfying

$$k_1 + k_2 + k_3 = 0, \quad \omega_1 + \omega_2 + \omega_3 = 0,$$

where the $\omega_j$ are the associated temporal wavenumbers. The amplitudes $A_j(t)$ of the three resonant sinusoidal waves $A_j(\varepsilon t)e^{i(k_j x+\omega_j t)}$ satisfy a closed system

$$\partial_\tau A_1 = i\gamma_1\overline{A_2 A_3}, \quad \partial_\tau A_2 = i\gamma_2\overline{A_1 A_3}, \quad \partial_\tau A_3 = i\gamma_3\overline{A_1 A_2}$$

with coefficients $\gamma_j = -\frac{1}{\omega_j}$. For this TWI system the subspace $M_1 = \{A_2 = A_3 = 0\}$ is invariant and its stability gives at least some hint about the long-time existence of solutions in the NLS-form. The stability of the subspace $M_1$ can be analyzed with the help of the energy

$$E = \varrho_2 A_2\overline{A_2} + \varrho_3 A_3\overline{A_3}$$

which can be chosen to be positive definite if $\varrho_2, \varrho_3$ have the same sign, which turns out to be possible if $\gamma_2$ and $\gamma_3$ have opposite signs.

In order to prove the approximation result in the stable case we construct a suitable energy which allows us to bound the error on the long time interval $[0, T_0/\varepsilon^2]$. This energy is based on the energy from the associated TWI systems combined with terms which allow us to eliminate the quadratic terms similar to the above-mentioned normal form transform.

For the sake of clarity we refrain from greatest generality and restrict ourselves to (1) as original system. As explained all principle difficulties which have to be overcome are covered by (1). It is the purpose of further research to apply the method to the water wave problem with non-zero surface tension, a relatively complicated quasilinear system which possesses the additional feature of a resonance at the wavenumber $k = 0$. This resonance is trivial for the water wave problem but a resonance at the wavenumber $k = 0$ always implies another resonance for the wavenumber $k = k_0$ which is in general non-trivial and especially it is non-trivial for the water wave problem. For the handling of this situation, in case of no other resonances, see [Schn98a]. We will not consider this much more complicated situation in this paper and refer to a forthcoming paper.

The plan of the paper is as follows. In order to explain our improvement we recall in Section 2 the existing proofs of the APP in case (i) of a pure cubic nonlinearity, i.e. $\alpha_q = 0$, and in case (ii) of a quadratic nonlinearity *without* non-trivial quadratic resonances, i.e. $\alpha_q \neq 0$, but $\alpha_r = 0$. For the rest of the paper, we are interested in the case of non-trivial quadratic resonances, i.e. $\alpha_q \neq 0$ and $\alpha_r \neq 0$. In Section 3, we prove the APP if the approximation is stable in the system for the TWI associated to the resonance. See Theorem 3.8. In Section 4.1, we construct a counterexample showing that the NLS equation can fail to approximate the original system if instability occurs for the approximation in the system for the TWI, i.e. we prove that non-trivial quadratic resonances are able to destroy solutions in NLS form even before the beginning of the natural time scale of the NLS equation. That means solutions of the original system (1) behave in this case differently than predicted by the NLS equation. In Section 4.2, we discuss to which extend our counterexample covers the general situation.

*Notations and some basic facts*: Fourier transform of a function $u$ is denoted with

$$(\mathcal{F}u)(k) = \hat{u}(k) = \frac{1}{2\pi}\int u(x)e^{-ikx}\,dx.$$

The point-wise multiplication $(uv)(x) = u(x)v(x)$ in $x$-space corresponds to the convolution

$$(\hat{u} * \hat{v})(k) = \int_{-\infty}^{\infty} \hat{u}(k-l)\hat{v}(l)\,dl$$

in Fourier space. Operators in Fourier space will be denoted with a $\hat{\cdot}$, in $x$-space without a $\hat{\cdot}$.

The Sobolev space $H^s$ is equipped with the norm

$$\|u\|_{H^s} = \left(\int |\hat{u}(k)|^2(1+|k|^2)^s\,dk\right)^{1/2}.$$

Moreover, let $\|u\|_{C_b^n} = \sum_{j=0}^{n} \|\partial_x^j u\|_{C_b^0}$, where $\|u\|_{C_b^0} = \sup_{x\in\mathbb{R}} |u(x)|$. We define the space $L^1(m)$ by $u \in L^1(m) \Leftrightarrow u\rho \in L^1$, where $\rho(k) = (1+k^2)^{1/2}$. We use the weighted Sobolev space $H^s(m)$ equipped with the norm $\|u\|_{H^s(m)} = \|u\rho^m\|_{H^s}$. We also use $L^2(m) = H^0(m)$.

We introduce the scaling operator $(S_\varepsilon u)(x) = u(\varepsilon x)$ and the translation operator $(\tau_y u)(x) = u(x+y)$. We have $\|S_\varepsilon A\|_{H^m} \leqslant C\varepsilon^{-1/2}\|A\|_{H^m}$, but $\|S_\varepsilon A\|_{C_b^m} \leqslant C\|A\|_{C_b^m}$. Moreover, $\mathcal{F}S_\varepsilon = \frac{1}{\varepsilon}S_{1/\varepsilon}\mathcal{F}$.

Fourier transform is an isomorphism from the weighted Sobolev space $H^n(m)(\mathbb{R},\mathbb{C})$ into the weighted Sobolev space $H^m(n)(\mathbb{R},\mathbb{C})$. It is a continuous mapping from $L^1(m)$ into $C_b^m$, but not vice versa. We have Sobolev's embedding theorem $\|u\|_{C_b^m} \leqslant C\|u\|_{H^s}$ if $m < s + \frac{1}{2}$.

We have $\|uv\|_{H^s(m)} \leqslant C\|u\|_{H^s(m)}\|v\|_{H^s(m)}$ for $s > \frac{1}{2}$ and $m \geqslant 0$ or $\|\hat{u} * \hat{v}\|_{H^s(m)} \leqslant C \|\hat{u}\|_{H^s(m)}\|\hat{v}\|_{H^s(m)}$ for $m > \frac{1}{2}$ and $s \geqslant 0$ due to Sobolev's embedding theorem.

## 2. Existing literature

As already said the proof of the APP for the NLS equation differs strongly in its difficulty depending on the properties of the nonlinearity and the linear dispersion relation. In order to explain our improvement we recall the existing results on cubic nonlinearities and quadratic nonlinearities without non-trivial quadratic resonances. If you are familiar with the existing literature you can skip the subsequent Section 2 completely and start directly with Section 3. Many arguments in Section 2 are only sketched. The detailed analysis can be found in Section 3.

### *2.1. Cubic nonlinearities*

In [KSM92], it was pointed out that the proof of the APP is more or less trivial if no quadratic terms are present in the nonlinearity. The simplest example fitting into this class is the nonlinear wave equation ($\alpha_q = \alpha_r = 0$, $\alpha_c = 1$ in (1))

$$\partial_t^2 u = \partial_x^2 u - u - u^3 \tag{6}$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}$ which can be written as a first-order system

$$\partial_t U = \Lambda U + C(U,U,U) \tag{7}$$

with $\Lambda$ a linear and $C(\cdot,\cdot,\cdot)$ a trilinear mapping. In detail, in Fourier space we have

$$\begin{aligned} S &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \\ \hat{\Lambda} &= \begin{pmatrix} i\sqrt{k^2+1} & 0 \\ 0 & -i\sqrt{k^2+1} \end{pmatrix}, \\ \hat{C}(\hat{U},\hat{U},\hat{U}) &= \frac{1}{\sqrt{k^2+1}} S^{-1} \tilde{C}(S\hat{U}, S\hat{U}, S\hat{U}), \\ \tilde{C}(\hat{U},\hat{V},\hat{W}) &= -\begin{pmatrix} 0 \\ \hat{U}_1 * \hat{V}_1 * \hat{W}_1 \end{pmatrix}, \end{aligned}$$

where $\hat{U} = (\hat{U}_1, \hat{U}_2)$. With the ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x + c_g t), \varepsilon^2 t) e^{i(k_0 x + \omega_0 t)} + \text{c.c.} + \mathcal{O}(\varepsilon^2) \tag{8}$$

with $k = k_0$ and $\omega = \omega_0$ satisfying the linear dispersion relation

$$\omega^2 = k^2 + 1, \tag{9}$$

the negative group velocity $c_{\mathrm{g}} = \partial_k \omega|_{k=k_0,\omega=\omega_0}$, and $0 \leqslant \varepsilon \ll 1$ a small perturbation parameter the NLS equation

$$2i\omega_0 \partial_T A = (1 - c_{\mathrm{g}}^2)\partial_X^2 A - 3A|A|^2 \tag{10}$$

with $T = \varepsilon^2 t$, $X = \varepsilon(x + c_{\mathrm{g}} t)$, and $A(X,T) \in \mathbb{C}$ can be derived by equating the coefficient in front of $\varepsilon^3 e^{i(k_0 x + \omega_0 t)}$ to zero. In order to justify (10) a solution $U$ is split into an approximation $\varepsilon\psi$ and into an error $\varepsilon^\beta R$ for a $\beta > 1$, i.e. $U = \varepsilon\psi + \varepsilon^\beta R$. Inserting this into (7) gives a system for the error

$$\partial_t R = \Lambda R + g(\varepsilon\psi, R) + \varepsilon^{-\beta}\,\mathrm{Res}(\varepsilon\psi) \tag{11}$$

with

$$\|g(\varepsilon\psi, R)\|_{\mathcal{H}^s} \leqslant C_1 \varepsilon^2 \|R\|_{\mathcal{H}^s} + C_2 \varepsilon^{\beta+1} \|R\|_{\mathcal{H}^s}^2$$

as long as $\|R(t)\|_{\mathcal{H}^s} \leqslant C_R$ with $C_R$ defined below, $\mathcal{H}^s = H^s \times H^s$ for $s \geqslant 1$ according to Sobolev's embedding theorem, and constants

$$C_1 = C_1(C_\psi), \quad C_2 = C_2(C_\psi, C_R), \tag{12}$$

where

$$C_\psi = \sup_{T \in [0,T_0]} \|A(T)\|_{H^{s_A}}$$

represents the bound on the approximation $\psi$ associated to a given solution

$$A \in C([0,T_0], H^{s_A}(\mathbb{R}, \mathbb{C}))$$

of the NLS equation (10) with $s_A \geqslant 0$ sufficiently big, determined below. By adding higher-order terms (see Section 3.2) to the approximation $\varepsilon\psi$ the so-called *residual*,

$$\mathrm{Res}(\varepsilon\psi) = -\partial_t U + \Lambda U + C(U,U,U),$$

i.e. the terms which do not drop out after inserting the approximation into the original system, can be made arbitrarily small, i.e. there exists a constant $C_3 = C_3(C_\psi, T_0, \beta) > 0$ such that for all $\varepsilon \in (0,1)$ we have

$$\sup_{\tau \in [0, T_0/\varepsilon^2]} e^{-\beta} \|\mathrm{Res}(\varepsilon\psi)(\tau)\|_{\mathcal{H}^s} \leqslant C_3 \varepsilon^2 \tag{13}$$

if $s_A - s \geqslant 3$. See Section 3.2. The linear operator $\Lambda$ generates an uniformly bounded strongly continuous semigroup $(e^{\Lambda t})_{t\geqslant 0}$ in $\mathcal{H}^s$ with $\|e^{\Lambda t}\|_{\mathcal{H}^s \to \mathcal{H}^s} \leqslant C_\Lambda$ for a constant $C_\Lambda > 0$ independent of $t \geqslant 0$. From the variation of constant formula

$$R(t) = \int_0^t e^{\Lambda(t-\tau)}(g(\psi, R) + e^{-\beta}\,\mathrm{Res}(\varepsilon\psi))(\tau)\,d\tau$$

we obtain for $S(t) = \sup_{\tau\in[0,t]} \|R(\tau)\|_{\mathcal{H}^s}$ that

$$S(t) \leqslant \int_0^t C_\Lambda(C_1\varepsilon^2 S(\tau) + C_2\varepsilon^{\beta+1}S(\tau)^2 + C_3\varepsilon^2)\,d\tau. \tag{14}$$

A simple application of Gronwall's inequality to (14) immediately shows that there exist constants $C_R > 0$, $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|R(t)\|_{\mathcal{H}^s} \leqslant C_R := T_0 C_\Lambda C_3 e^{C_\Lambda(C_1+1)T_0}, \tag{15}$$

independent of $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 > 0$ had to be chosen so small that

$$\varepsilon_0^{\beta-1} C_2(C_\psi, C_R)C_R \leqslant 1. \tag{16}$$

In summary, to a given $C_R = C_R(T_0, C_\Lambda, C_1, C_3)$ defined in (15) we have a $C_2$ by (12) and to that $C_2$ we have an $\varepsilon_0 > 0$ by (16). Hence, the APP holds and so there are solutions $u$ of (6) which behave for all $t \in [0, T_0/\varepsilon^2]$ as predicted by the NLS equation (10). The formulation (APP) from the introduction follows from (15) by applying Sobolev's embedding theorem.

### *2.2. Quadratic nonlinearities without non-trivial quadratic resonances*

At a first sight the above proof completely fails if quadratic terms are present in the nonlinearity. The simplest example fitting into this class is the nonlinear wave equation ($\alpha_r = \alpha_c = 0$, $\alpha_q = 1$ in (1))

$$\partial_t^2 u = \partial_x^2 u - u + u^2 \tag{17}$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}$ which can be written as a first-order system

$$\partial_t U = \Lambda U + B(U, U) \tag{18}$$

with $\Lambda$ a linear and $B(\cdot,\cdot)$ a bilinear mapping. In detail, in Fourier space we have now

$$\begin{aligned}
S &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \\
\hat{\Lambda} &= \begin{pmatrix} i\sqrt{k^2+1} & 0 \\ 0 & -i\sqrt{k^2+1} \end{pmatrix}, \\
\hat{B}(\hat{U},\hat{U}) &= \frac{1}{\sqrt{k^2+1}} S^{-1}\tilde{B}(S\hat{U}, S\hat{U}), \\
\tilde{B}(\hat{U},\hat{V}) &= \begin{pmatrix} 0 \\ \hat{U}_1 * \hat{V}_1 \end{pmatrix},
\end{aligned}$$

where $\hat{U} = (\hat{U}_1, \hat{U}_2)$. Inserting the ansatz $U = \varepsilon\psi + \varepsilon^{\beta} R$ for a $\beta > 2$, with $\varepsilon\psi$ again a modified approximation in the sense of (13), into (18) gives a system

$$\partial_t R = \Lambda R + 2\varepsilon B(\psi, R) + \mathcal{O}(\varepsilon^2),$$

where the term $2\varepsilon B(\psi, R)$ allows us not to proceed as above. However, this term turned out to be oscillatory in time and can be eliminated with averaging or a normal form transformation. This observation goes back to Kalyakin [Kal88].

For simplicity, we proceed as in [Sh85] and use the fact that for the nonlinear wave equation (17) more general all quadratic terms $B(U,U)$ can be eliminated. In order to do so we make a near identity change of variables

$$U = w + Q(w, w) \tag{19}$$

with $Q$ an autonomous bilinear mapping. This gives

$$\partial_t w + Q(\partial_t w, w) + Q(w, \partial_t w) = \Lambda w + \Lambda Q(w, w) + B(w, w) + \mathcal{O}(\|w\|^3)$$

and so

$$\partial_t w = \Lambda w + \Lambda Q(w, w) - Q(\Lambda w, w) - Q(w, \Lambda w) + B(w, w) + \mathcal{O}(\|w\|^3).$$

In order to eliminate the quadratic terms $B(w, w)$ we have to find a $Q$ such that

$$\Lambda Q(w, w) - Q(\Lambda w, w) - Q(w, \Lambda w) + B(w, w) = 0.$$

In Fourier space with

$$
\begin{aligned}
(\hat{B}(\hat{w},\hat{w}))_j &= \sum_{m,n=\pm} \int (1+k^2)^{-1/2} \hat{b}^j_{mn}(k,k-l,l)\hat{w}_m(k-l)\hat{w}_n(l)\,dl, \\
(\hat{Q}(\hat{w},\hat{w}))_j &= \sum_{m,n=\pm} \int \hat{q}^j_{mn}(k,k-l,l)\hat{w}_m(k-l)\hat{w}_n(l)\,dl
\end{aligned}
$$

for the $j$th component of $B$ and $Q$, $\omega_\pm(k) = \pm i\sqrt{1+k^2}$ and $\hat{b}^j_{mn}, \hat{q}^j_{mn}$ some coefficients we obtain the well-known relation

$$
i(1+k^2)^{1/2}(\omega_j(k)-\omega_m(k-l)-\omega_n(l))\hat{q}^j_{mn}(k,k-l,l) = \hat{b}^j_{mn}(k,k-l,l). \tag{20}
$$

For notational simplicity we included the term $(1+k^2)^{-1/2}$ in the definition of $\hat{b}$. Eq. (20) can be resolved with respect to $\hat{q}^h_{mn}$ due to the validity of the non-resonance condition

$$
\inf_{j,m,n\in\{+,-\},\ k,l\in\mathbb{R}} |(1+k^2)^{1/2}(\omega_j(k)-\omega_m(k-l)-\omega_n(l))| \geqslant 1. \tag{21}
$$

Since $\sup_{j,m,n\in\{+,-\},\ k,l\in\mathbb{R}} |\hat{b}^j_{mn}(k,k-l,l)| \leqslant C < \infty$ we obtain

$$
\|(Q(w,w))\|_{\mathcal{H}^s} \leqslant C\|w\|_{\mathcal{H}^s}\|w\|_{\mathcal{H}^s}.
$$

For more details see Section 3. Thus, the transformation (19) can be resolved with respect to $w$ for $\|w\|_{\mathcal{H}^s} > 0$ sufficiently small. Therefore (18) transforms into

$$
\partial_t w = \Lambda w + N_3(w) \tag{22}
$$

with a nonlinearity $N_3$ which does not contain quadratic terms anymore and so the proof of the APP from Section 2.1 then applies line for line to (22).

**Remark 2.1.** Since only the term $2\varepsilon B(\psi, R)$ has to be eliminated, since the approximation $\varepsilon\psi$ is of order $\mathcal{O}(\varepsilon)$ only close to the wavenumbers $\pm k_0$, since all kernels are globally Lipschitz, and since we have a real-valued problem the non-resonance condition (21) can be relaxed to

$$
\inf_{j,n\in\{+,-\},\ k\in\mathbb{R}} |(\omega_j(k)-\omega_+(k_0)-\omega_n(k-k_0))| > 0. \tag{23}
$$

**Remark 2.2.** The idea of eliminating the quadratic terms can also be used for the justification of other modulation equations [SU01]. In [SU01], at three places the term $i\sqrt{1+k^2}$ has been forgotten.

## 3. The error estimates

For the rest of the paper we are interested in the case of non-trivial quadratic resonances, i.e. $\alpha_r \neq 0$ and $\alpha_q \neq 0$ in (1). It is the purpose of this section to prove the APP under the additional assumption that the NLS approximation is stable in the systems for the TWI associated to the resonances. See Theorem 3.8. Since cubic terms will not lead to any difficulties in justifying the NLS equation we will restrict ourselves in the following to the system ($\alpha_q = 1$, $\alpha_r = 1$, $\alpha_c = 0$ in (1))

$$\partial_t^2 u = \partial_x^2 u - u - \partial_x^4 u + u^2 \tag{24}$$

with $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, and $t \geqslant 0$ as original system. We comment at the end of the proof on the case $\alpha_c \neq 0$. We proceed as follows. In Section 3.1, we derive the NLS equation followed by a section in which we construct an approximation $\varepsilon\psi$ such that the residual $\mathrm{Res}(\varepsilon\psi)$ is at least of order $\mathcal{O}(\varepsilon^{2+\beta})$ for a $\beta > 2$. In Section 3.3, we analyze the set of resonances and the associated TWI systems. In Section 3.4, we construct a suitable energy which allows us to bound the error on the long time interval $[0, T_0/\varepsilon^2]$. This energy is based on the energy from the associated TWI systems combined with terms which allow us to eliminate the quadratic terms similar to the normal form transform of Section 2.2. Finally, in Section 3.5 we summarize our result.

### *3.1. Derivation of the NLS equation*

In the following two sections, we construct an approximation $\varepsilon\psi$ such that the residual $\mathrm{Res}(\varepsilon\psi)$ is at least of order $\mathcal{O}(\varepsilon^{2+\beta})$ for a $\beta > 2$. We recall this well-known construction in order to figure out the necessary non-resonance conditions.

As is Section 2.2, Eq. (24) can be written as a first-order system

$$\partial_t U = \Lambda U + B(U,U) \tag{25}$$

with $\Lambda$ a linear, and $B$ a bilinear mapping. In detail, in Fourier space we have as in the sections before

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad \hat{\Lambda}(k) = \begin{pmatrix} i\omega_+(k) & 0 \\ 0 & i\omega_-(k) \end{pmatrix},$$

$$\hat{B}(\hat{U},\hat{V}) = \frac{1}{\omega_+} S^{-1}\tilde{B}(S\hat{U}, S\hat{V}), \qquad \tilde{B}(\hat{U},\hat{V}) = \begin{pmatrix} 0 \\ \hat{U}_1 * \hat{V}_1 \end{pmatrix},$$

where the eigenvalues $i\omega_\pm$ are now given by

$$i\omega_\pm(k) = \pm i\sqrt{1 + k^2 + k^4}.$$

We introduce now the coefficients $\hat{b}^j_{mn}(k, k-l, l)$ of the bilinear mapping $B$ by

$$(\hat{B}(\hat{u}, \hat{v})(k))_j = \int \sum_{m,n\in\{+,-\}} \hat{b}^j_{mn}(k, k-l, l)\hat{u}_m(k-l)\hat{v}_n(l)\, dl,$$

such that (25) can be written as

$$\partial_t \hat{U}_+(k,t) = i\omega_+(k)\hat{U}_+(k,t) + \int \sum_{m,n\in\{+,-\}} \hat{b}^+_{mn}(k, k-l, l)\hat{U}_m(k-l,t)\hat{U}_n(l,t)\, dl,$$

$$\partial_t \hat{U}_-(k,t) = i\omega_-(k)\hat{U}_-(k,t) + \int \sum_{m,n\in\{+,-\}} \hat{b}^-_{mn}(k, k-l, l)\hat{U}_m(k-l,t)\hat{U}_n(l,t)\, dl.$$

In $x$-space this is a pseudo-differential equation. With $e_j$ we denote the $j$th basis vector of $\mathbb{R}^2$. Without loss of generality we derive the NLS equation for the first component, and so in $x$-space we make the ansatz

$$U = \varepsilon\psi_1 + \varepsilon\psi_{-1} + \varepsilon^2\psi_2 + \varepsilon^2\psi_0 + \varepsilon^2\psi_{-2} \tag{26}$$

with

$$\begin{aligned}
(\varepsilon\psi_1)_+ &= \varepsilon A\left(\varepsilon(x + c_{\mathrm{g}}t), \varepsilon^2 t\right) e^{i(k_0x+\omega_0 t)},\\
(\varepsilon\psi_1)_- &= 0,\\
(\varepsilon\psi_{-1})_+ &= 0,\\
(\varepsilon\psi_{-1})_- &= \overline{(\varepsilon\psi_1)_+},\\
(\varepsilon^2\psi_2)_j &= \varepsilon^2 A_{2j}\left(\varepsilon(x + c_{\mathrm{g}}t), \varepsilon^2 t\right) e^{2i(k_0x+\omega_0 t)},\\
(\varepsilon^2\psi_0)_j &= \varepsilon^2 A_{0j}\left(\varepsilon(x + c_{\mathrm{g}}t), \varepsilon^2 t\right)
\end{aligned}$$

and $(\psi_{-j_2})_+ = \overline{(\psi_{j_2})_-}$ for $j_2 \geqslant 1$. Equating the coefficients of $\varepsilon^{j_1} e^{ij_2(k_0x+\omega_0 t)} e_+$ to zero gives for

$$\begin{aligned}
(j_1, j_2) = (1,1): \quad i\omega_0 &= i\omega_+(k_0),\\
(j_1, j_2) = (2,1): c_{\mathrm{g}}\partial_X A &= i\partial_k\omega_+|_{k=k_0}(-i\partial_X A),\\
(j_1, j_2) = (3,1): \quad \partial_T A &= -i\partial_k^2\omega_+|_{k=k_0}\partial_X^2 A/2 + 2\textstyle\sum_{j\in\{+,-\}} \hat{b}^+_{+j}(1,1,0)AA_{0j}\\
&\quad + 2\textstyle\sum_{j\in\{+,-\}} \hat{b}^+_{+j}(1,-1,2)\overline{A}A_{2j}.
\end{aligned}$$

We obtain for $j \in \{+,-\}$ and

$$
\begin{aligned}
(j_1, j_2) = (2,0):\qquad\qquad 0 &= i\omega_j|_{k=0} A_{0j} + 2\hat{b}^j_{++}(0,1,-1)A\overline{A},\\
(j_1, j_2) = (2,2):\ 2i\omega_0 A_{2j} &= i\omega_j|_{k=2k_0} A_{2j} + 2\hat{b}^j_{++}(2,1,1)A^2.
\end{aligned}
$$

Eliminating $A_{0j}$ and $A_{2j}$ in the equation for $(j_1, j_2) = (3,1)$ by the algebraic relations for $(j_1, j_2) = (2,0)$ and $(j_1, j_2) = (2,2)$ gives the NLS equation

$$
\partial_T A = -i\partial_k^2 \omega_+|_{k=k_0} \partial_X^2 A/2 + \gamma A^2 \overline{A} \tag{27}
$$

with

$$
\gamma = -2 \sum_{j\in\{+,-\}} \left( \frac{\hat{b}^+_{+j}(1,1,0)\hat{b}^j_{++}(0,1,-1)}{i\omega_j|_{k=0}} + \frac{\hat{b}^+_{+j}(1,-1,2)\hat{b}^j_{++}(2,1,1)}{-2i\omega_0 + i\omega_j|_{k=2k_0}} \right).
$$

Hence, in order to derive the NLS equation there should by no quadratic resonances of the basic wavenumber $k_0$ with the wavenumbers 0 and $2k_0$, i.e. we need
(B1) For $j \in \{+,-\}$ we assume $\omega_j|_{k=0} \neq 0$ and $2\omega_+|_{k=k_0} \neq \omega_j|_{k=2k_0}$.

Hence, the NLS equation can be derived under much weaker conditions than the non-resonance condition (5) used in the proof of the APP in [Kal88]. Therefore, the question occurs, is the NLS equation also valid if (5) is not satisfied.

In the derivation we used for instance the formal relation

$$
\begin{aligned}
\omega_+(-i\partial_x)(A(\varepsilon x)e^{ij_2k_0x}) &= \int \omega_+(k)\frac{1}{\varepsilon}\hat{A}\left(\frac{k-j_2k_0}{\varepsilon}\right)e^{ikx}\,dk\\
&= \int \omega_+(j_2k_0 + \varepsilon K)\hat{A}(K)e^{ij_2k_0x + i\varepsilon Kx}\,dK\\
&= \int (\omega_+(j_2k_0) + \mathcal{O}(\varepsilon))\hat{A}(K)e^{i\varepsilon Kx}\,dK \cdot e^{ij_2k_0x}\\
&= \omega_+(j_2k_0)A(\varepsilon x)e^{ij_2k_0x} + \mathcal{O}(\varepsilon)
\end{aligned}
$$

which leads further with

$$
\omega_+(j_2k_0 + \varepsilon K) = \omega_+(j_2k_0) + \varepsilon\omega_+'(j_2k_0)K + \varepsilon^2\omega_+''(j_2k_0)K^2/2 + \mathcal{O}(\varepsilon^3)
$$

to

$$
\begin{aligned}
\omega_+(-i\partial_x)(A(\varepsilon x)e^{ij_2k_0x}) = \Big[&\omega_+(j_2k_0) - i\varepsilon\omega_+'(j_2k_0)\partial_X\\
&- \tfrac{1}{2}\varepsilon^2\omega_+''(j_2k_0)\partial_X^2 + \mathcal{O}(\varepsilon^3))A(X)\Big]\, e^{ij_2k_0x}.
\end{aligned}
$$

Similar relations are used for the bilinear terms.

*3.2. The improved approximation and estimates for the residual*

The first step in proving that the NLS equation (27) provides a good approximation for solutions of the original system (25) are estimates for the residual

$$\mathrm{Res}(u) = -\partial_t u + \Lambda u + B(u,u),$$

i.e. for the terms which do not cancel after inserting the approximation into the original system (25), cf. [CSS92]. The residual is a measure how much $u$ fails to be a solution of (25). It turned out that in order to show the APP it is advantageous to make the residual smaller by adding higher-order terms to the approximation (26).

Therefore, we make the ansatz

$$\varepsilon\psi = \sum_{|j_2|<M} \sum_{\beta(j_2,j_1)\leqslant M} \varepsilon^{\beta(j_2,j_1)} \psi_{j_2}^{j_1} \tag{28}$$

with

$$\begin{aligned} \beta(j_2,j_1) &= 1 + ||j_2| - 1| + j_1, \\ \psi_{j_2}^{0} &= \psi_{j_2} \quad \text{from above}, \\ \left(\psi_{j_2}^{j_1}\right)_j &= A_{j_2 j}^{j_1}(\varepsilon(x + c_{\mathrm{g}} t), \varepsilon^2 t) e^{i j_2 (k_0 x + \omega_0 t)}. \end{aligned}$$

Inserting this into (25) and equating the coefficients of $\varepsilon^{\beta(j_2,j_1)} e^{i j_2 (k_0 x + \omega_0 t)} e_j$ to zero gives for not $((j_2,j) = (1,+)$ and $(j_2,j) = (-1,-))$ a system

$$\left(j_2\omega_0 + \omega_j(j_2 k_0) A_{j_2 j}^{j_1}\right) = g_{j_2 j}^{j_1},$$

where the $g_{j_2 j}^{j_1}$ only contains $A_{\tilde{j}_2 \tilde{j}}^{\tilde{j}_1}$ $\left(\text{and derivatives of } A_{\tilde{j}_2 \tilde{j}}^{\tilde{j}_1}\right)$ with $\tilde{j}_1 < j_1$ of if $j_1 = \tilde{j}_1$ then $\tilde{j}_2 < j_2$ or $\tilde{j}_2 = \pm 1$.

For $(j_2,j) = (1,+)$ and $(j_2,j) = (-1,-)$ we obtain

$$\partial_T A_{j_2 j}^{j_1} = i v_1 \partial_X^2 A_{j_2 j}^{j_1} + g_{j_2 j}^{j_1}$$

with $g_{j_2 j}^{j_1}$ being affine in $A_{j_2 j}^{j_1}$ only containing $A_{\tilde{j}_2 \tilde{j}}^{\tilde{j}_1}$ $\left(\text{and derivatives of } A_{\tilde{j}_2 \tilde{j}}^{\tilde{j}_1}\right)$ with $\tilde{j}_1 < j_1$.

Hence, for a given solution $A \in C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$ of the NLS equation (27) we can compute for $(j_2,j) = (1,+)$ and $(j_2,-) = (-1,-)$ that

$$A_{j_2 j}^{j_1} \in C([0,T_0], \;\; H^{s_A - 2 - j_1}(\mathbb{R},\mathbb{C}))$$

and for not $((j_2, j) = (1, +)$ and $(j_2, j) = (-1, -))$ that

$$A^{j_1}_{j_2 j} \in C([0, T_0], \quad H^{s_A - j_1}(\mathbb{R}, \mathbb{C}))$$

if the non-resonance condition (B2) is satisfied. The loss of regularity is according to the derivatives in the $g^{j_1}_{j_2 j}$.

(B2) For given $M$ assume for all $j_2 \leqslant M$ with $j_2 \notin \{-1, 1\}$ and $j \in \{+, -\}$ that $-j_2\omega_+(k_0) + \omega_j(j_2 k_0) \neq 0$.

This assumption can be relaxed in the sense that if $(-j_2\omega_+(k_0) + \omega_j(j_2k_0)) = 0$ then assume $g^{j_1}_{j_2 j}(j_2 k_0) = 0$, too. See [Kal88, Assumption IV].

Using the following lemma and similar estimates for the nonlinear terms allows us to estimate the residual rigorously.

**Lemma 3.1.** *For all* $s \geqslant 0$, $m \in \mathbb{N}$, $C_1 > 0$ *there exists a* $C_2 > 0$ *such that for all* $\varepsilon \in (0, 1)$ *the following holds. Let* $\mu : \mathbb{R} \to \mathbb{C}$ *with* $|\mu(k)| \leqslant C_1|k|^m$ *and* $A \in H^{s+m}(\mathbb{R}, \mathbb{C})$. *Then*

$$\|\mu(-i\partial_x)A(\varepsilon\cdot)\|_{H^s} \leqslant C_2\varepsilon^{m-\frac{1}{2}}\|A\|_{H^{s+m}}.$$

**Proof.** The proof uses that $|\mu(k)| = \mathcal{O}(|k|^m)$ for $|k| \to 0$ and the concentration of the Fourier modes of $A(\varepsilon\cdot)$ at $k = 0$. In detail,

$$\begin{aligned}
\|\mu(-i\partial_x)A(\varepsilon\cdot)\|_{H^s} &\leqslant C \left\| \mu(\cdot)\frac{1}{\varepsilon}\hat{A}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(s)} \\
&\leqslant C\varepsilon^{-\frac{1}{2}}\|\mu(\varepsilon\cdot)\hat{A}(\cdot)\|_{L^2(s)} \\
&\leqslant C\varepsilon^{-\frac{1}{2}} \sup_{k\in\mathbb{R}} \left| \frac{C_1|k|^m\varepsilon^m}{(1+k^2)^{m/2}}\right| \cdot \|\hat{A}\|_{L^2(s+m)} \\
&\leqslant C_2\varepsilon^{m-\frac{1}{2}}\|A\|_{H^{m+s}}. \qquad \square
\end{aligned}$$

Applying Lemma 3.1 for instance to

$$|\mu(k)| = |\omega_+(j_2k_0 + k) - (\omega_+(j_2k_0) + \omega_+'(j_2k_0)k + \omega_+''(j_2k_0)k^2/2)| \leqslant C|k|^3$$

finally shows

**Lemma 3.2.** *Consider* (25) *under the validity of the assumptions* (B1) *and* (B2) *for a given M. Then for all* $C_A, T_0 > 0, s \geqslant 1$ *there exist* $C_{\mathrm{Res}}, C_\Psi, \varepsilon_1 > 0$ *such that the following holds for all* $\varepsilon \in (0, \varepsilon_1)$:

*Let* $A \in C([0,T_0], H^{s+M+4}(\mathbb{R},\mathbb{C}))$ *be a solution of the NLS equation* (27) *with*

$$\sup_{T\in[0,T_0]} \|A(T)\|_{H^{s+M+4}} \leqslant C_A.$$

*Then the approximation* $\psi$ *defined in* (28) *exists for all* $T \in [0, T_0]$ *and satisfies*

$$\begin{aligned} \sup_{T\in[0.T_0]} \|\hat{\psi}_{j_2}^{j_1}(T)\|_{L^1(s)} &\leqslant C_\Psi, \\ \sup_{T\in[0,T_0]} \|\varepsilon\psi(T) - \varepsilon\psi_{NLS}(T)\|_{C_b^s} &\leqslant C_\Psi \varepsilon^{3/2}, \\ \sup_{T\in(0,T_0]} \|\mathrm{Res}(\varepsilon\psi(T))\|_{H^s} &\leqslant C_{\mathrm{Res}}\varepsilon^{M+\frac{1}{2}}. \end{aligned}$$

**Remark 3.3.** By doing the approximation in a more clever way, i.e. by choosing compact support in Fourier space, it is possible to reduce the regularity for the solutions of the NLS equation from $H^{s+M+4}$ to $H^{\max(s,M+4)}$. See for instance [Schn98a].

**Remark 3.4.** The first estimate is used for instance for the estimate

$$\|B(\psi, R)\|_{H^s} \leqslant C\|\psi\|_{C_b^s}\|R\|_{H^s} \leqslant \|\hat{\psi}\|_{L^1(s)}\|R\|_{H^s}.$$

Note that $\|\hat{\psi}\|_{H^s} = \mathcal{O}(\varepsilon^{-1/2})$.

### *3.3. Analysis of the resonances, the TWI system*

A wavenumber $\tilde{k}$ is called resonant to the basic wavenumber $k_0$ if

$$r_{m,n}(\tilde{k}) = \omega_m(\tilde{k}) - \omega_+(k_0) - \omega_n(\tilde{k} - k_0) = 0$$

for some $m, n \in \{+, -\}$. Fig. 2 shows that for given $k_0 > 0$ there is at least one pair of resonances. For $k_0 > 0$ sufficiently large there are up to three pair of resonances.

**Remark 3.5.** According to the fact that we have a real-valued problem we always have $\omega_+(k) = -\omega_-(-k)$. A consequence is for instance that

$$\omega_+(k) - \omega_+(k_0) - \omega_-(k - k_0) = 0$$

implies

$$-\omega_-(-k) - \omega_+(k_0) - \omega_-(k - k_0) = 0.$$

According to the last remark assume now that we have a pair of resonances $k_2, k_3$ resonant to $k_1 = k_0$ satisfying

$$k_1 + k_2 + k_3 = 0, \quad \omega_1 + \omega_2 + \omega_3 = 0,$$

where the $\omega_j$ are the associated temporal wavenumbers. With the ansatz

$$\begin{aligned} U \;=\; & \varepsilon A_1(\varepsilon t)e^{i(k_1 x+\omega_1 t)}e_{j_1} + \varepsilon A_2(\varepsilon t)e^{i(k_2 x+\omega_2 t)}e_{j_2} \\ & + \varepsilon A_3(\varepsilon t)e^{i(k_3 x+\omega_3 t)}e_{j_3} + \text{c.c.} \end{aligned}$$

by equating the coefficients of $\varepsilon^2 e^{i(k_1 x+\omega_1 t)}e_{j_1}$, $\varepsilon^2 e^{i(k_2 x+\omega_2 t)}e_{j_2}$, and $\varepsilon^2 e^{i(k_3 x+\omega_3 t)}e_{j_3}$ to zero, a closed system

$$\left.\begin{aligned} \partial_\tau A_1 &= i\gamma_1 \overline{A_2 A_3}, \\ \partial_\tau A_2 &= i\gamma_2 \overline{A_1 A_3}, \\ \partial_\tau A_3 &= i\gamma_3 \overline{A_1 A_2} \end{aligned}\right\} \tag{29}$$

with coefficients $\gamma_j = -\frac{1}{\omega_j}$ for the $A_j$ is obtained.

This system possesses three invariant subspaces consisting of fixed points, namely

$$M_1 = \{A_2 = A_3 = 0\},\; M_2 = \{A_1 = A_3 = 0\},\; M_3 = \{A_1 = A_2 = 0\}.$$

In order to compute the stability of the subspace $M_1$ we consider the energy

$$E = \varrho_2 A_2 \overline{A_2} + \varrho_3 A_3 \overline{A_3}$$

and obtain

$$\begin{aligned} \frac{d}{d\tau}E \;&=\; \varrho_3 A_3 \partial_\tau \overline{A_3} + \varrho_3 \overline{A_3}\partial_\tau A_3 + \varrho_2 A_2 \partial_\tau \overline{A_2} + \varrho_2 \overline{A_2}\partial_\tau A_2 \\ &=\; \varrho_3 A_3(-i\gamma_3 A_1 A_2) + \varrho_3 \overline{A_3}(i\gamma_3 \overline{A_1 A_2}) \\ &\quad + \varrho_2 A_2(-i\gamma_2 A_1 A_3) + \varrho_2 \overline{A_2}(i\gamma_2 A_1 \overline{A_3}) \\ &=\; -i(\varrho_3\gamma_3 + \varrho_2\gamma_2)(A_2 A_3 A_1 - \overline{A_2 A_3 A_1}). \end{aligned}$$

We can choose a positive definite energy, i.e. $\varrho_2, \varrho_3$ the same sign, if $\gamma_2$ and $\gamma_3$ have opposite signs, i.e. then $\varrho_3\gamma_3 + \varrho_2\gamma_2 = 0$ such that

$$\frac{d}{d\tau}E = 0.$$

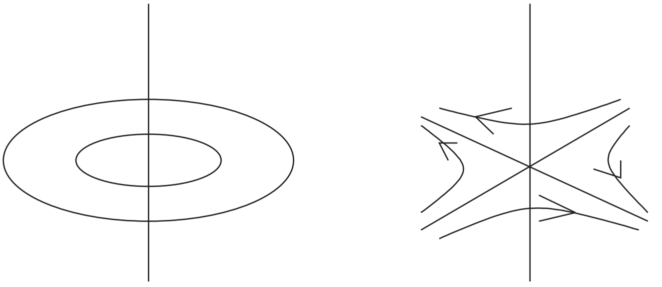

Fig. 3. Stability (line of centers) and instability (line of saddle points) of the space $M_1$ in case $\gamma_2\gamma_3 < 0$ and $\gamma_2\gamma_3 > 0$, respectively.

Thus, in case that $\gamma_2$ and $\gamma_3$ have opposite signs, the subspace $M_1$ is stable. If $\gamma_2$ and $\gamma_3$ have the same sign the subspace $M_1$ is unstable according to the linearized system

$$\partial_\tau^2 A_2 = \gamma_2\gamma_3 |A_1|^2 A_2.$$

See Fig. 3.

Since $\omega_1 + \omega_2 + \omega_3 = 0$, for (1) always one subspace of the $M_j$'s is unstable and two subspaces are stable. This is reminiscent of the rotation of a rigid body.

We will prove now that the APP holds if the subspace $M_1$ is stable in all TWI systems associated to the possible resonances $k$. In Section 4.1, we construct a counterexample showing that the APP not necessarily holds if $M_1$ is unstable in at least one of the TWI systems (29) associated to the resonances $k$.

### 3.4. The energy estimates

As in Section 2, a solution $U$ of (25) is written as a sum of the approximation $\varepsilon\psi$ and an error $\varepsilon^\beta R$, i.e.

$$U = \varepsilon\psi + \varepsilon^\beta R \tag{30}$$

for a $\beta > 2$. Inserting (30) into (25) gives

$$\partial_t R = \Lambda R + 2\varepsilon B(\psi, R) + \varepsilon^\beta B(R, R) + \varepsilon^{-\beta}\,\mathrm{Res}(\varepsilon\psi). \tag{31}$$

In order to estimate $R$ we use an energy

$$\mathcal{E}_s(R, R) = \sum_{j=0}^{s} E_j(R, R) + \varepsilon Q_j(\psi, R, R)$$

with $E_s$ a symmetric bilinear and $Q_s$ a symmetric trilinear mapping with

$$\hat{E}_s(\hat{R},\hat{R}) = \int \sum_{j\in\{+,-\}} \hat{\sigma}_j(k)(ik)^s \hat{R}_j(k)\overline{(ik)^s \hat{R}_j(k)}\,dk$$

and

$$\begin{aligned}\hat{Q}_s(\hat{\psi},\hat{R},\hat{R}) = {} & \iint \sum_{j,m,n\in\{+,-\}} \overline{\hat{q}^j_{mn}(k,k-l,l)}(ik)^s\hat{R}_j(k)\overline{(ik)^s\hat{\psi}_m(k-l)\hat{R}_n(l)}\,dl\,dk \\ & + \iint \sum_{j,m,n\in\{+,-\}} \hat{q}^j_{mn}(k,k-l,l)\overline{(ik)^s\hat{R}_j(k)}(ik)^s\hat{\psi}_m(k-l)\hat{R}_n(l)\,dl\,dk.\end{aligned}$$

With the coefficient $\hat{\sigma}_j(k)$ the first term $E_s(R,R)$ can be adjusted to the energy of the TWI system associated to the resonances. The coefficient $\hat{q}^j_{mn}(k,k-l,l)$ will be chosen in such a way that outside of the resonances in the time derivative of the energy all terms associated to the terms of order $\mathcal{O}(\varepsilon)$ in (31) are eliminated similar to the normal form transform from Section 2. In order to explain the idea to control the $\mathcal{O}(\varepsilon)$ terms in (31) we start with $s=0$. In the following we use the abbreviations $\mathcal{E}=\mathcal{E}_0$, $E=E_0$, and $Q=Q_0$.

We obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(R,R) = {} & E(\partial_t R,R)+E(R,\partial_t R) \\ & +\varepsilon Q(\partial_t\psi,R,R)+\varepsilon Q(\psi,\partial_t R,R)+\varepsilon Q(\psi,R,\partial_t R) \\ = {} & E(\Lambda R,R)+E(R,\Lambda R)+2\varepsilon E(R,B(\psi,R))+2\varepsilon E(B(\psi,R),R) \\ & +\varepsilon^\beta E(R,B(R,R))+\varepsilon^\beta E(B(R,R),R) \\ & +\varepsilon^{-\beta}E(R,\mathrm{Res}(\varepsilon\psi))+\varepsilon^{-\beta}E(\mathrm{Res}(\varepsilon\psi),R) \\ & +\varepsilon Q(\partial_t\psi,R,R)+\varepsilon Q(\psi,\partial_t R,R)+\varepsilon Q(\psi,R,\partial_t R) \\ = {} & \varepsilon(2E(R,B(\psi,R))+2E(B(\psi,R),R) \\ & +Q(\Lambda\psi,R,R)+Q(\psi,\Lambda R,R)+Q(\psi,R,\Lambda R))+\varepsilon^2 H_1\end{aligned}$$

with $H_1=\mathcal{O}(1)$. We used $E(\Lambda R,R)+E(R,\Lambda R)=0$ due to the skew-symmetry of the operator $\Lambda$ and $\partial_t\psi=\Lambda\psi+\mathcal{O}(\varepsilon)$. We are done if we can eliminate the order $\mathcal{O}(\varepsilon)$-terms. In order to do so we have to find a $Q$ such that

$$\begin{aligned}& Q(\Lambda\psi,R,R)+Q(\psi,\Lambda R,R)+Q(\psi,R,\Lambda R) \\ & \quad +2E(R,B(\psi,R))+2E(B(\psi,R),R)=0.\end{aligned} \tag{32}$$

We have

$$
\begin{aligned}
&\hat{E}(\hat{R}, \hat{B}(\hat{\psi}, \hat{R})) + \hat{E}(\hat{B}(\hat{\psi}, \hat{R}), \hat{R}) \\
&\quad = \int\!\!\int \sum_{j,m,n \in \{+,-\}} \hat{\sigma}_j(k) \hat{R}_j(k) \overline{\hat{b}^j_{mn}(k, k-l, l) \hat{\psi}_m(k-l) \hat{R}_n(l)}\, dl\, dk \\
&\qquad + \int\!\!\int \sum_{j,m,n \in \{+,-\}} \hat{\sigma}_j(k) \overline{\hat{R}_j(k)} \hat{b}^j_{mn}(k, k-l, l) \hat{\psi}_m(k-l) \hat{R}_n(l)\, dl\, dk
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{Q}(\Lambda\hat{\psi}, \hat{R}, \hat{R}) + \hat{Q}(\hat{\psi}, \Lambda\hat{R}, \hat{R}) + \hat{Q}(\hat{\psi}, \hat{R}, \Lambda\hat{R}) \\
&\quad = i \int\!\!\int \sum_{j,m,n \in \{+,-\}} (\omega_j(k) - \omega_m(k-l) - \omega_n(l)) \\
&\qquad \times \overline{\hat{q}^j_{mn}(k, k-l, l)} \hat{R}_j(k) \overline{\hat{\psi}_m(k-l) \hat{R}_n(l)}\, dl\, dk \\
&\qquad - i \int\!\!\int \sum_{j,m,n \in \{+,-\}} (\omega_j(k) - \omega_m(k-l) - \omega_n(l)) \\
&\qquad \times \hat{q}^j_{mn}(k, k-l, l) \overline{\hat{R}_j(k)} \hat{\psi}_m(k-l) \hat{R}_n(l)\, dl\, dk.
\end{aligned}
$$

The approximation is of the form

$$
\varepsilon\hat{\psi}(k) = \varepsilon\hat{a}_+(k-k_0)e_+ + \varepsilon\hat{a}_-(k+k_0)e_- + \mathcal{O}(\varepsilon^2)
$$

with

$$
\begin{aligned}
\hat{a}_+(k-k_0) &= \varepsilon^{-1}\hat{A}\left(\frac{k-k_0}{\varepsilon}\right) e^{i\omega_+(k_0)t} e^{i\omega'_+(k_0)(k-k_0)t}, \\
\hat{a}_-(k+k_0) &= \varepsilon^{-1}\hat{\overline{A}}\left(\frac{k+k_0}{\varepsilon}\right) e^{i\omega_-(-k_0)t} e^{i\omega'_-(-k_0)(k+k_0)t}.
\end{aligned}
$$

Inserting this into (32) yields

$$
\begin{aligned}
&2 \int\!\!\int \sum_{j,n \in \{+,-\}} \hat{\sigma}_j(k) \hat{R}_j(k) \overline{\hat{b}^j_{+n}(k, k-l, l) \hat{a}_+(k-l-k_0) \hat{R}_n(l)}\, dl\, dk \\
&\quad + 2 \int\!\!\int \sum_{j,n \in \{+,-\}} \hat{\sigma}_j(k) \hat{R}_j(k) \overline{\hat{b}^j_{-n}(k, k-l, l) \hat{a}_-(k-l+k_0) \hat{R}_n(l)}\, dl\, dk
\end{aligned}
$$

$$
\begin{aligned}
&+2\iint \sum_{j,n\in\{+,-\}} \hat{\sigma}_j(k)\overline{\hat{R}_j(k)}\hat{b}^j_{+n}(k,k-l,l)\hat{a}_+(k-l-k_0)\hat{R}_n(l)\,dl\,dk \\
&+2\iint \sum_{j,n\in\{+,-\}} \hat{\sigma}_j(k)\overline{\hat{R}_j(k)}\hat{b}^j_{-n}(k,k-l,l)\hat{a}_-(k-l+k_0)\hat{R}_n(l)\,dl\,dk \\
&+i\iint \sum_{j,n\in\{+,-\}} (\omega_j(k)-\omega_+(k-l)-\omega_n(l)) \\
&\times \overline{\hat{q}^j_{+n}(k,k-l,l)}\hat{R}_j(k)\overline{\hat{a}_+(k-l-k_0)\hat{R}_n(l)}\,dl\,dk \\
&+i\iint \sum_{j,n\in\{+,-\}} (\omega_j(k)-\omega_-(k-l)-\omega_n(l)) \\
&\times \overline{\hat{q}^j_{-n}(k,k-l,l)}\hat{R}_j(k)\overline{\hat{a}_-(k-l+k_0)\hat{R}_n(l)}\,dl\,dk \\
&-i\iint \sum_{j,n\in\{+,-\}} (\omega_j(k)-\omega_+(k-l)-\omega_n(l)) \\
&\times \hat{q}^j_{+n}(k,k-l,l)\overline{\hat{R}_j(k)}\hat{a}_+(k-l-k_0)\hat{R}_n(l)\,dl\,dk \\
&-i\iint \sum_{j,n\in\{+,-\}} (\omega_j(k)-\omega_-(k-l)-\omega_n(l)) \\
&\times \hat{q}^j_{-n}(k,k-l,l)\overline{\hat{R}_j(k)}\hat{a}_-(k-l+k_0)\hat{R}_n(l)\,dl\,dk = 0.
\end{aligned}
$$

Using the following lemma shows that the coefficient functions which so far depend on the two variables $k$ and $l$ can be written as a function of the variable $k$ alone. The lemma will be applied to the above expression using that $\hat{\psi}$ is only of order $\mathcal{O}(1)$ close to the basic wavenumbers $\pm k_0$.

**Lemma 3.6.** *Fix $m \geqslant 1$, $k_0 \in \mathbb{R}$, and let $\tilde{K} = \tilde{K}(k,l) \in C^2(\mathbb{R}^2, \mathbb{C})$ be globally Lipschitz continuous. Then there exists a $C > 0$ such that*

$$
\begin{aligned}
&\bigg\| \int \tilde{K}(\cdot,l)\hat{R}_1(\cdot)\hat{a}\left(\frac{\cdot - l - k_0}{\varepsilon}\right)\hat{R}_2(l)\,dl \\
&\quad - \int \tilde{K}(\cdot,\cdot - k_0)\hat{R}_1(\cdot)\hat{a}\left(\frac{\cdot - l - k_0}{\varepsilon}\right)\hat{R}_2(l)\,dl \bigg\|_{L^2(m)} \\
&\quad \leqslant C\varepsilon \|\hat{R}_1\|_{L^2(m)}\|\hat{R}_2\|_{L^2(m)}\|\hat{a}\|_{L^1(m+1)}.
\end{aligned}
$$

**Proof.** The right-hand side is estimated as follows:

$$\begin{aligned}
&\left\| \int (\tilde{K}(\cdot, l) - \tilde{K}(\cdot, \cdot - k_0)) \hat{R}_1(\cdot) \hat{a} \left( \frac{\cdot - l - k_0}{\varepsilon} \right) \hat{R}_2(l)\, dl \right\|_{L^2(m)} \\
&\leqslant C \left\| \int \left| (\cdot - l - k_0) \hat{R}_1(\cdot) \hat{a} \left( \frac{\cdot - l - k_0}{\varepsilon} \right) \hat{R}_2(l) \right| dl \right\|_{L^2(m)} \\
&\leqslant C \|\hat{R}_1\|_{L^2(m)} \|\hat{R}_2\|_{L^2(m)} \left\| (\cdot - k_0) \hat{a} \left( \frac{\cdot - k_0}{\varepsilon} \right) \right\|_{L^1(m)} \\
&\leqslant C\varepsilon \|\hat{R}_1\|_{L^2(m)} \|\hat{R}_2\|_{L^2(m)} \|\hat{a}\|_{L^1(m+1)}
\end{aligned}$$

by Young's inequality and Sobolev's embedding theorem. □

**Remark 3.7.** The estimate is only used for $k$ close to the resonances. Therefore, the assumption of global Lipschitz continuity of $\tilde{K}$ will trivially be satisfied.

Outside a neighborhood of the non-resonant wavenumbers and $|k - l \pm k_0| < \delta$ for a $\delta > 0$ independent of $0 \leqslant \varepsilon \ll 1$ we choose $\sigma_j = 1$ and

$$\begin{aligned}
\hat{q}^j_{+n}(k, k-l, l) &= 2i(\omega_j(k) - \omega_+(k-l) - \omega_n(l))^{-1} \hat{b}^j_{+n}(k, k-l, l), \\
\hat{q}^j_{-n}(k, k-l, l) &= 2i(\omega_j(k) - \omega_-(k-l) - \omega_n(l))^{-1} \hat{b}^j_{-n}(k, k-l, l).
\end{aligned}$$

Inside the neighborhood of the resonant wavenumbers we apply Lemma 3.6 and so (32) transforms into the question of finding $\sigma_j$'s and $q^j_{mn}$'s satisfying

$$\begin{aligned}
&2 \iint \sum_{j,n \in \{+,-\}} \hat{\sigma}_j(k) \hat{R}_j(k) \overline{\hat{b}^j_{+n}(k) \hat{a}_+(k - l - k_0) \hat{R}_n(l)}\, dl\, dk \\
&\quad + 2 \iint \sum_{j,n \in \{+,-\}} \hat{\sigma}_j(k) \hat{R}_j(k) \overline{\hat{b}^j_{-n}(k) \hat{a}_-(k - l + k_0) \hat{R}_n(l)}\, dl\, dk \\
&\quad + 2 \iint \sum_{j,n \in \{+,-\}} \hat{\sigma}_j(k) \overline{\hat{R}_j(k)} \hat{b}^j_{+n}(k) \hat{a}_+(k - l - k_0) \hat{R}_n(l)\, dl\, dk \\
&\quad + 2 \iint \sum_{j,n \in \{+,-\}} \hat{\sigma}_j(k) \overline{\hat{R}_j(k)} \hat{b}^j_{-n}(k) \hat{a}_-(k - l + k_0) \hat{R}_n(l)\, dl\, dk \\
&\quad + i \iint \sum_{j,n \in \{+,-\}} \hat{L}^j_{+n}(k) \overline{\hat{q}^j_{+n}(k)} \hat{R}_j(k) \overline{\hat{a}_+(k - l - k_0) \hat{R}_n(l)}\, dl\, dk \\
&\quad + i \iint \sum_{j,n \in \{+,-\}} \hat{L}^j_{-n}(k) \overline{\hat{q}^j_{-n}(k)} \hat{R}_j(k) \overline{\hat{a}_-(k - l + k_0) \hat{R}_n(l)}\, dl\, dk
\end{aligned}$$

$$
-i \iint \sum_{j,n \in \{+,-\}} \hat{L}^{j}_{+n}(k) \hat{q}^{j}_{+n}(k) \overline{\hat{R}_j(k)} \hat{a}_+(k-l-k_0) \hat{R}_n(l)\, dl\, dk
$$

$$
-i \iint \sum_{j,n \in \{+,-\}} \hat{L}^{j}_{-n}(k) \hat{q}^{j}_{-n}(k) \overline{\hat{R}_j(k)} \hat{a}_-(k-l+k_0) \hat{R}_n(l)\, dl\, dk = 0,
$$

where we used the abbreviations

$$
\begin{aligned}
\hat{L}^{j}_{\pm n}(k) &= \omega_j(k) - \omega_\pm(\pm k_0) - \omega_n(k \mp k_0),\\
\hat{b}^{j}_{\pm n}(k) &= \hat{b}^{j}_{\pm n}(k, \pm k_0, k \pm k_0),\\
\hat{q}^{j}_{\pm n}(k) &= \hat{q}^{j}_{\pm n}(k, \pm k_0, k \pm k_0).
\end{aligned}
$$

Assume for simplicity that there is only one pair of resonant wavenumbers, namely $\tilde{k}$ and $\tilde{k} - k_0$, i.e. w.l.o.g. for our purposes

$$
\hat{L}^{+}_{++}(\tilde{k}) = \omega_+(\tilde{k}) - \omega_+(k_0) - \omega_+(\tilde{k} - k_0) = 0
$$

which implies

$$
\hat{L}^{+}_{-+}(\tilde{k} - k_0) = \omega_+(\tilde{k} - k_0) - \omega_-(-k_0) - \omega_+(\tilde{k}) = 0.
$$

In the following, these resonant wavenumbers are denoted by $k_2 = \tilde{k}$ and $k_3 = \tilde{k} - k_0$. For the neighborhood close to $k = k_2$ we introduce the new coordinates $k = k_2 + r$ and $l = k_3 + s$. For the neighborhood close to $k = k_3$ we introduce the new coordinates $k = k_3 + s$ and $l = k_2 + r$. The problem then consists in finding $\hat{q}^{j}_{mn}$'s and $\hat{\sigma}_j$'s such that

$$
\begin{aligned}
&2 \iint \hat{\sigma}_+(k_2 + r) \hat{R}_+(k_2 + r) \overline{\hat{b}^{+}_{++}(k_2 + r) \hat{a}_+(r - s) \hat{R}_+(k_3 + s)}\, ds\, dr\\
&\quad + \iint i \hat{L}^{+}_{++}(k_2 + r) \overline{\hat{q}^{+}_{++}(k_2 + r)} \hat{R}_+(k_2 + r) \overline{\hat{a}(r - s) \hat{R}_+(k_3 + s)}\, ds\, dr\\
&\quad + 2 \iint \hat{\sigma}_+(k_3 + s) \overline{\hat{R}_+(k_3 + s)} \hat{b}^{+}_{-+}(k_3 + s) \hat{a}_-(s - r) \hat{R}_+(k_2 + r)\, dr\, ds\\
&\quad - \iint i \hat{L}^{+}_{-+}(k_3 + s) \hat{q}^{+}_{-+}(k_3 + s) \overline{\hat{R}_+(k_3 + s)} \hat{a}_-(s - r) \hat{R}_+(k_2 + r)\, dr\, ds + \text{c.c.f}=0,
\end{aligned}
$$

where c.c.f stands for the Fourier transform of the complex conjugate function in $x$-space. Using $\hat{a}_-(s - r) = \overline{\hat{a}_+(r - s)}$ and equating the coefficient in front of

$$
\hat{R}_+(k_2 + r) \overline{\hat{a}_+(r - s) \hat{R}_+(k_3 + s)}
$$

to zero yields

$$\begin{aligned} &2\hat{\sigma}_+(k_2+r)\overline{\hat{b}^+_{++}(k_2+r)} + i\hat{L}^+_{++}(k_2+r)\overline{\hat{q}^+_{++}(k_2+r)} \\ &\quad + 2\hat{\sigma}_+(k_3+s)\hat{b}^+_{-+}(k_3+s) - i\hat{L}^+_{-+}(k_3+s)\hat{q}^+_{-+}(k_3+s) = 0. \end{aligned}$$

By assumption we have $\overline{\hat{b}^+_{++}(k_2)}\hat{b}^+_{-+}(k_3) < 0$. Choose now $\hat{\sigma}_+ \in \mathbb{R}$ such that

$$\hat{\sigma}_+(k_2)\overline{\hat{b}^+_{++}(k_2)} + \hat{\sigma}_+(k_3)\hat{b}^+_{-+}(k_3) = 0$$

similar as we did for the associated TWI system (29). Let

$$\begin{aligned} g(k_2+r) &= 2\hat{\sigma}_+(k_2+r)\left(\hat{b}^+_{++}(k_2+r) - \hat{b}^+_{++}(k_2)\right), \\ g(k_3+s) &= 2\hat{\sigma}_+(k_3+s)\left(\hat{b}^+_{-+}(k_3+s) - \hat{b}^+_{-+}(k_3)\right). \end{aligned}$$

Choose then

$$\begin{aligned} \hat{q}^+_{++}(k_2+r) &= i\left(\hat{L}^+_{++}(k_2+r)\right)^{-1} g(k_2+r), \\ \hat{q}^+_{++}(k_3+s) &= i\left(\hat{L}^+_{-+}(k_3+s)\right)^{-1} g(k_3+s). \end{aligned}$$

According to the fact that

$$|g(k_2+r)| \leqslant C|r| \quad \text{and} \quad |g(k_3+s)| \leqslant C|s|$$

and since $\partial_k \hat{L}^+_{-+}(k_3) \neq 0$ and $\partial_k \hat{L}^+_{++}(k_2) \neq 0$ we have the boundedness of $\hat{q}^+_{++}$ near $k_2$ and of $\hat{q}^+_{-+}$ near $k_3$. At all other places we set $\hat{\sigma}_j = 1$, and $\hat{q}^j_{mn} = 0$. Therefore, we are done with eliminating the order $\mathcal{O}(\varepsilon)$-terms in the time derivative of the energy, i.e. we find that all terms of order $\mathcal{O}(\varepsilon)$ cancel in $\frac{d}{dt}\mathcal{E}_0(R,R)$. It is an easy exercise to prove that by exactly the same procedure all terms of order $\mathcal{O}(\varepsilon)$ cancel in $\frac{d}{dt}\mathcal{E}_j(R,R)$ for all $0 \leqslant j \leqslant s$.

Due to the boundedness of the $\hat{\sigma}_j$ and the $\hat{q}^j_{mn}$ the scalar product $\mathcal{E}_s(\cdot,\cdot)$ is equivalent to the usual $H^s$ scalar product, i.e. there exist positive constants $c_1$ and $c_2$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\|R\|^2_{H^s} \leqslant c_1 \mathcal{E}_s(R,R) \leqslant c_2 \|R\|^2_{H^s}. \tag{33}$$

Therefore, we can sum up our above estimates and obtain with constants

$$C_1 = C_1(C_\Psi, C_{\mathrm{Res}}, c_j), \quad C_2 = C_2(C_\Psi, C_{\mathrm{Res}}, C_R, c_j), \quad C_3 = C_3(C_\Psi, C_{\mathrm{Res}}, c_j),$$

that

$$\begin{aligned}\tfrac{1}{2}\partial_t\mathcal{E}_s(R,R) &\leqslant \varepsilon^2 C_1\mathcal{E}_s(R,R)+\varepsilon^{\beta}C_2\mathcal{E}_s(R,R)^{3/2}+(\varepsilon^2C_3)\mathcal{E}_s(R,R)^{1/2}\\ &\leqslant \varepsilon^2 C_1\mathcal{E}_s(R,R)+\varepsilon^{\beta}C_2\mathcal{E}_s(R,R)^{3/2}\\ &\quad+(C_3\varepsilon^2)+(\varepsilon^2C_3)\mathcal{E}_s(R,R),\end{aligned}\tag{34}$$

i.e. for $y(T)=\mathcal{E}_s(R(t),R(t))$ with $T=\varepsilon^2 t$ we obtain the ordinary differential equation

$$\begin{aligned}\dot{y} &= (C_1+C_3)y+\varepsilon^{\beta}C_2y^{3/2}+C_3\\ &\leqslant (C_1+C_3+1)y+C_3.\end{aligned}$$

Applying Gronwall's inequality shows for all $T\in[0,T_0/\varepsilon^2]$ that

$$y(T)=(y(0)+C_3T_0)e^{(C_1+C_3+1)T_0}=:C_R,$$

where we have chosen $\varepsilon_0>0$ so small that $\varepsilon^{\beta-2}C_2(C_R)(c_2C_R)\leqslant 1$. This yields

$$\sup_{t\in[0,T_0/\varepsilon^2]}\|R(t)\|_{H^s}^2\leqslant c_1C_R.$$

Therefore, we are done.

The case $\alpha_c\neq 0$ works exactly the same. Since normal form transforms for the elimination of the quadratic terms or the inclusion of these normal form terms into the energy do not affect the order of the higher-order terms the final energy estimates (34) remain unchanged in the case $\alpha_c\neq 0$. This follows by an easy counting of powers of $\varepsilon$ as in Section 2.1.

### *3.5. The result*

We proved

**Theorem 3.8.** *Consider* (1) *with* $\alpha_r=1$, $\alpha_q=1$, *and* $\alpha_c=0$. *Choose a wavenumber* $k_0>0$ *for which* (B1) *and* (B2) *should hold. Assume further that the NLS subspaces in the TWI systems associated to the resonances are stable. Moreover, let* $s_A-s\geqslant 8$ *and* $s\geqslant 1$. *Let* $A\in C([0,T_0],H^{s_A}(\mathbb{R},\mathbb{C}))$ *be a solution of the NLS equation* (4). *Then there exist* $\varepsilon_0>0$ *and* $C>0$ *such that for all* $\varepsilon\in(0,\varepsilon_0)$ *there are solutions* $u$ *of* (1) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]}\|u(\cdot,t)-\varepsilon\psi_{\mathrm{NLS}}(\varepsilon,\cdot,t)\|_{H^s}\leqslant C\varepsilon^{3/2}.$$

**Remark 3.9.** The APP as stated in the introduction follows immediately with Sobolev's embedding theorem.

## 4. Failure of the APP

In the unstable case the situation is less clear. In Section 4.1, we construct a counterexample showing that the APP does not necessarily holds if instability occurs for the NLS approximation in the system for the TWI, i.e. we prove that non-trivial quadratic resonances are able to destroy solutions in NLS form before the end of the natural time scale of the NLS equation. This means that solutions of the original system (1) behave in this case differently than predicted by the NLS equation. In detail, for $2\pi$-spatially periodic solutions of a scalar pseudo-differential equation. we prove that the NLS approximation breaks down after a time scale $\mathcal{O}(\varepsilon^{-1}|\ln\varepsilon|)$ which is much smaller than the natural time scale $\mathcal{O}(\varepsilon^{-2})$ of the NLS approximation. In Section 4.2, we give some arguments why we expect that the APP also holds for spatially localized solutions of the NLS equation, although the instability is present. Moreover, we explain why we do not expect the validity of the APP for non-localized solutions of the NLS equation. Again, although we restrict ourselves to a nonlinear wave equation as original system we believe that the statements hold in more general situations, too.

### *4.1. A counterexample in case of periodic boundary conditions*

For notational simplicity we consider a real-valued scalar pseudo-differential equation which is given in Fourier space by

$$\partial_t \hat{u}(k,t) = i\omega(k)\hat{u}(k,t) + \rho(k)(\hat{u} * \hat{u})(k,t),$$

where the eigenvalues $i\omega$ of the linearized problem satisfy $\omega(k) = -\omega(-k)$. Moreover, we have $\rho(k) = -\rho(k)$ and for the solution $\hat{u}(k) = \overline{\hat{u}(-k)}$. In $x$-space we consider $2\pi$-spatially periodic solutions of the scalar pseudo-differential equation, i.e. in Fourier space $k \in \mathbb{Z}$. Hence, $\hat{u}(0,t) = \hat{u}(0,0)$ which is assumed to be zero. We make the ansatz

$$u(x,t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} u_k(t)e^{ikx}$$

and find the system for the $u_k$'s. For the eigenvalues $i\omega_k = i\omega(k)$ of this system we assume the resonance

$$\omega_1 + \omega_4 + \omega_{-5} = 0$$

and that except of the complex conjugate resonance no other resonances should exist. We can then apply the near identity normal form transform from above in order to eliminate all quadratic terms except of the ones according to the resonance, i.e. for the new variables $v_k$ we obtain a system

$$\begin{aligned}
\partial_t v_1 &= i\omega_1 v_1 + i\gamma_1\overline{v_{-5}v_4} + \mathcal{O}(|v|^3),\\
\partial_t v_4 &= i\omega_4 v_4 + i\gamma_4\overline{v_{-5}v_1} + \mathcal{O}(|v|^3),\\
\partial_t v_{-5} &= i\omega_{-5}\overline{v_{-5}} + i\gamma_{-5}\overline{v_1 v_4} + \mathcal{O}(|v|^3),\\
\partial_t v_r &= i\omega_r v_r + \mathcal{O}(|v|^3)
\end{aligned}$$

with similar equations for $v_{-1}$, $v_{-4}$, and $v_5$, where $v_r$ stands for all other modes except of these six, and where $v$ stands for all modes, i.e. $\mathcal{O}(|v|^3)$ represents all terms which are at least cubic in the $v_k$. Moreover we have $\gamma_k = \rho(k)$. For this system we derive a NLS equation for the basic wavenumber $k_0 = 1$ by making the ansatz $v_1 = \varepsilon A_1(\varepsilon^2 t)e^{i\omega_1 t}$, $v_k = 0$ for $k \in \mathbb{N} \setminus \{1\}$ and $v_{-k} = \overline{v_k}$ for $k \in \mathbb{N}$. In this case, the usual NLS equation degenerates to the ordinary differential equation

$$\partial_t A_1 = i\nu_2 A_1 |A_1|^2$$

with $\nu_2 \in \mathbb{R}$ computed as before. Necessary for an approximation theorem is at least the fact that all modes $v_k$ for $k \neq \pm 1$ are strictly smaller than $\mathcal{O}(\varepsilon)$, i.e. $\mathcal{O}(\varepsilon^\rho)$ for an $\rho > 1$ on the time interval of the approximation which is of order $\mathcal{O}(1/\varepsilon^2)$. Hence, we introduce $w_k$ by $v_k = \varepsilon w_k$ and obtain for the equations we are interested in that

$$\begin{aligned}
\partial_t w_1 &= i\omega_1 w_1 + i\varepsilon\gamma_1 \overline{w_{-5} w_4} + \mathcal{O}(\varepsilon^2),\\
\partial_t w_4 &= i\omega_4 w_4 + i\varepsilon\gamma_4 \overline{w_{-5} w_1} + \mathcal{O}(\varepsilon^2),\\
\partial_t w_{-5} &= i\omega_{-5} \overline{w_{-5}} + i\varepsilon\gamma_{-5} \overline{v_1 w_4} + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

We argue by contradiction and assume that the APP holds. Then we have the $\mathcal{O}(\varepsilon^2)$-boundedness of all terms indicated with $\mathcal{O}(\varepsilon^2)$ uniformly in time on an $\mathcal{O}(1/\varepsilon^2)$ time interval. In order to prove that the APP does not hold we will show that on a time scale of order $\mathcal{O}(\varepsilon^{-1}|\ln\varepsilon|) \ll \mathcal{O}(\varepsilon^{-2})$ we find $w_4$ and $w_{-5}$ to be of order $\mathcal{O}(1)$. In order to do so we set $w_j(t) = e^{i\omega_j t} B_j(\varepsilon t)$ and find

$$\begin{aligned}
\partial_\tau B_1 &= i\gamma_1 \overline{B_{-5} B_4} + \mathcal{O}(\varepsilon),\\
\partial_\tau B_4 &= i\gamma_4 \overline{B_{-5} B_1} + \mathcal{O}(\varepsilon),\\
\partial_\tau B_{-5} &= i\gamma_{-5} \overline{B_1 B_4} + \mathcal{O}(\varepsilon).
\end{aligned}$$

where we rescaled time $\tau = \varepsilon t$. Ignoring the $\mathcal{O}(\varepsilon)$ terms which are now uniformly bounded with this order on an time interval of length $\mathcal{O}(1/\varepsilon)$ in $\tau$ gives the TWI system associated to the resonance. It has been analyzed before. It is considered here under the additional assumption that $\gamma_4\gamma_{-5} > 0$ and under initial conditions $0 \neq B_1 = \mathcal{O}(1)$, $B_4 = \mathcal{O}(\varepsilon^\rho)$, and $B_{-5} = \mathcal{O}(\varepsilon^\rho)$ for a $\rho \geqslant 1$. Under these assumptions the $B_1$-subspace, i.e. $\{B_4 = B_{-5} = 0\}$, is unstable, and the variables $B_4$ and $B_{-5}$ grow exponentially along the unstable manifold with an $\mathcal{O}(1)$ growth rate. See Fig. 3. Consequently, $B_4$ and $B_{-5}$ are of order $\mathcal{O}(1)$ at a time of order $\mathcal{O}(\varepsilon^{-1}|\ln\varepsilon|)$ in $t$ or $\mathcal{O}(|\ln\varepsilon|)$ in $\tau$. It is an easy exercise to prove that until this time $B_1$ is still of order $\mathcal{O}(1)$, may be by making the time a little bit smaller (independent of $\varepsilon$). Moreover, the $\mathcal{O}(\varepsilon)$ terms do not have any influence on the phase picture on this relatively short time scale. Therefore, we are done. It is obvious that the arguments also work for the nonlinear wave equation (1) by adjusting $k_0$, $\alpha_\mathrm{r}$, and $\alpha_\mathrm{q}$ properly.

### 4.2. Some heuristics for the general situation

The purpose of this section is to discuss the general situation in case of an unstable resonance if the periodic boundary conditions are skipped. Depending on the group velocities at the resonant wavenumbers and on the spatial localization of the NLS solutions we make three claims about the validity or the failure of the NLS approximation.

In general, the resonance is not an integer multiple of the basic wavenumber $k_0$. Therefore, $2\pi$ spatially periodic solutions no longer can be considered to prove a failure of the APP. Then the associated TWI system (29) will be a set of PDEs, namely

$$\left.\begin{aligned} \partial_\tau A_1 &= c_1 \partial_X A_1 + i\gamma_1 \overline{A_2 A_3}, \\ \partial_\tau A_2 &= c_2 \partial_X A_2 + i\gamma_2 \overline{A_1 A_3}, \\ \partial_\tau A_3 &= c_3 \partial_X A_3 + i\gamma_3 \overline{A_1 A_2}, \end{aligned}\right\} \tag{35}$$

where $A_j = A_j(X, T)$ and where $c_j = \partial_k \omega|_{k=k_j,\omega=\omega_j}$. It is obtained by the ansatz

$$\begin{aligned} U &= \varepsilon A_1(\varepsilon x, \varepsilon t) e^{i(k_1 x + \omega_1 t)} e_1 + \varepsilon A_2(\varepsilon x, \varepsilon t) e^{i(k_2 x + \omega_2 t)} e_{j_1} \\ &\quad + \varepsilon A_3(\varepsilon x, \varepsilon t) e^{i(k_3 x + \omega_3 t)} e_{j_2} + \text{c.c.} \end{aligned}$$

The initial magnitude of $A_2$ and $A_3$ depend on the regularity of $A_1|_{\tau=0}$, the initial condition of the NLS equation. More or less, if $A_1|_{\tau=0}$ is $s_A$-times differentiable, then we have $A_j|_{\tau=0} = \mathcal{O}(\varepsilon^{s_A})$ for $j = 2, 3$. If $A_1|_{\tau=0}$ is analytic in a strip of width $2r$ around the real axis in the complex plane, then we have $A_j|_{\tau=0} = \mathcal{O}(\exp(-r/\varepsilon))$ for $j = 2, 3$. This has been used in [Schn98b] to obtain an $\mathcal{O}(1/\varepsilon^2)$ time scale in $t$ for the approximations. We refer to the same paper for the justification of the last statements. In order to prove the breakdown of the approximation long before the $\mathcal{O}(1/\varepsilon^2)$ time scale in $t$ we have to prove that $A_2$ and $A_3$ will be $\mathcal{O}(1)$ long before the $\mathcal{O}(1/\varepsilon^2)$ time scale in $t$ or the $\mathcal{O}(1/\varepsilon)$ time scale in $\tau$. According to the fact that we are unable to make rigorous statements in the general situation we will argue with (35) in the following. At the end of this section we comment on this procedure.

We will consider three cases.

(A) In the not very generic case that the velocities $c_1 = c_2 = c_3 = c$ are the same by chance, by the coordinate transform $X \mapsto X + c\tau$ the transport terms $c_j \partial_X A_j$ can be eliminated and we are back to the original TWI system (29). Thus we make the following claim.

**Claim 4.1.** *The APP does not hold if $M_1$, the subspace associated to the NLS approximation in the TWI systems* (29), *is unstable in at least one of the TWI systems* (29) *associated to the resonances k in the* (*non-generic*) *case that all velocities $c_j$ for $j = 1, 2, 3$ are the same.*

We have to make this more precise. There is a set of initial conditions for which we expect that the approximation result should hold and at least one suitable norm, for instance the $L^2$-norm, in which the error is of the same order than the approximation and the solution long before an $\mathcal{O}(1/\varepsilon^2)$-time scale.

(B) Next we consider the case $c_2 = c_3 \neq c_1$. Again by a coordinate transform we can assume that $c_2 = c_3 = 0$. In order to keep the notational complexity at a reasonable level we assume w.l.o.g. for our purposes $\gamma_2 = \gamma_3 = 1$, such that the $A_1$-subspace is unstable. For the same reason we assume $c_1 = 1$. Since, initially $A_2, A_3 = \mathcal{O}(\varepsilon^{s_A})$ in lowest order $A_1$ is only translated in space, i.e. $A_1(X,\tau) = A(X+\tau)$. Similar to Section 3 we have

$$\partial_\tau \left( \int |A_2|^2 - |A_3|^2 \right) dx = 0.$$

To simplify further, we choose $A$ to be a real function and $A_2 = A_3$. The last choice is respected by the equations, neglecting the nonlinear terms in the $A_1$ equation, as we already did. Then the reduced system is given by

$$\partial_\tau A_2 = iA\overline{A_2}.$$

We split $A_2$ in real and imaginary part, i.e. $A_2 = A_\mathrm{r} + iA_\mathrm{i}$ and find

$$\partial_\tau A_\mathrm{r} = AA_\mathrm{i}, \quad \partial_\tau A_\mathrm{i} = AA_\mathrm{r}$$

such that

$$\partial_\tau (A_\mathrm{r} - A_\mathrm{i}) = -A(A_\mathrm{r} - A_\mathrm{i}), \quad \partial_\tau (A_\mathrm{r} + A_\mathrm{i}) = A(A_\mathrm{r} + A_\mathrm{i}).$$

Hence we finally consider the reduced system for $r = A_\mathrm{r} + A_\mathrm{i}$, namely

$$\partial_\tau r(X,\tau) = A(X+\tau) r(X,\tau).$$

(i) Our first choice is a strongly localized function $A$, namely

$$A(\xi) = \begin{cases} 1, & \xi \in [0,1], \\ 0, & \text{else}. \end{cases}$$

It is easy to see that in this case

$$|r(X,\tau)| \leqslant e|r(X,0)|$$

for all $X \in \mathbb{R}$ and all $\tau \geqslant 0$.

(ii) Our second choice is a function which is not that strongly localized, namely

$$A(\xi) = \min(1, |\xi|^{-\alpha})$$

for $\alpha \geqslant 0$. For $\alpha = 0$ which corresponds to the counterexample in Section 4.1 we already know that the APP does not hold. We consider the point $X = 1$ and write in the following $r(\tau)$ for $r(1, \tau)$. There we have to solve the ODE

$$\partial_\tau r = \tau^{-\alpha} r$$

which is solved by ($\alpha \neq 1$)

$$r(\tau) = r(1) \exp((1-\alpha)^{-1}(\tau^{1-\alpha} - 1)).$$

It is very easy to see that for $\alpha \in (0, 1)$ solutions to initial condition $r(1) = \mathcal{O}(\varepsilon^{s_A})$ become $\mathcal{O}(1)$ on a time scale $\mathcal{O}(|\ln \varepsilon|) \ll \mathcal{O}(\varepsilon^{-1})$ in $\tau$. However, for $\alpha > 1$ the solutions stay $\mathcal{O}(\varepsilon^{s_A})$-bounded for all $\tau \geqslant 1$. The transport term $\partial_X A_1$ will not affect the instabilities on an $\mathcal{O}(1)$ time scale in $\tau$. However, on a longer time scale $A_1$ will be transported and so $A_2$ and $A_3$ have not enough time to grow to a reasonable size in $L^\infty$ if the velocities are not the same.

(C) We do not expect that things change in the general case, i.e. also $c_2 \neq c_3$. Therefore, we make the following two claims. Our first claim includes general $H^s$-functions, i.e. natural initial data for the NLS equation.

**Claim 4.2.** *The APP does not hold if $M_1$ is unstable in at least one of the TWI systems* (29) *associated to the resonances k under the assumption that $A_1|_{T=0}$ is not spatially localized, i.e. that $X \mapsto A_1|_{T=0}(X)$ is not in $L^1$.*

However, our next claim is

**Claim 4.3.** *If the solutions $A_1$ of the NLS equation are spatially localized, i.e. decay sufficiently fast for $|x| \to \infty$ then the APP does hold although $M_1$ is unstable in at least one of the TWI systems* (29) *associated to the resonances k.*

By looking at the mode distribution of the solutions of the original system after the normal form transform the modes at the resonant wavenumbers are concentrated in an $\mathcal{O}(\varepsilon)$ neighborhood at these wavenumbers. Therefore, the consideration of (35) somehow makes sense at least before $A_2$ and $A_3$ have grown to a reasonable size. It is the purpose of future research to make these claims rigorous or to disprove the claimed statements.

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## References


[AS81] M.J. Ablowitz, H. Segur, Solitons and the inverse scattering transform, SIAM Studies in Applied Mathematics, vol. 4, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1981.

[CSS92] W. Craig, C. Sulem, P.L. Sulem, Nonlinear modulation of gravity waves: a rigorous approach, Nonlinearity 5 (1992) 497–552.

[Kal88] L.A. Kalyakin, Asymptotic decay of a one-dimensional wave packet in a nonlinear dispersive medium, Math. USSR Sb. Surveys 60 (2) (1988) 457–483.

[KSM92] P. Kirrmann, G. Schneider, A. Mielke, The validity of modulation equations for extended systems with cubic nonlinearities, Proc. Roy. Soc. Edinburgh 122A (1992) 85–91.

[Schn95] G. Schneider, Validity and limitation of the Newell–Whitehead equation, Math. Nachr. 176 (1995) 249–263.

[Schn98a] G. Schneider, Approximation of the Korteweg–de Vries equation by the Nonlinear Schrödinger equation, J. Differential Equations 147 (1998) 333–354.

[Schn98b] G. Schneider, Justification of modulation equations for hyperbolic systems via normal forms, Nonlinear Differential Equations Appl. (NODEA) 5 (1998) 69–82.

[SU01] G. Schneider, H. Uecker, Nonlinear coupled mode dynamics in hyperbolic and parabolic periodically structured spatially extended systems, Asymptotic Anal. 28 (2) (2001) 163–180.

[Sh85] J. Shatah, Normal forms and quadratic nonlinear Klein–Gordon equations, Comm. Pure Appl. Math. 38 (1985) 685–696.

[Za68] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, Sov. Phys. J. Appl. Mech. Tech. Phys. 4 (1968) 190–194.